

Algebraic Structures of $N=(4,4)$ and $N=(8,8)$ SUSY Sigma Models on Lie groups and SUSY WZW Models

M. Aali-Javanangrouh^{a *}, A. Rezaei-Aghdam^{a †}

^a *Department of Physics, Faculty of science, Azarbaijan Shahid Madani University
53714-161, Tabriz, Iran*

February 25, 2014

Abstract

In this study, algebraic structures of the $N = (4, 4)$ and $N = (8, 8)$ SUSY two dimensional sigma models on Lie groups (in general) and the $N = (4, 4)$ and $N = (8, 8)$ SUSY WZW models (in special) are obtained. These algebraic structures are reduced to the Lie bialgebraic structures as for the $N = (2, 2)$ case; with the difference that there is a one 2-cocycle for the $N = (4, 4)$ case and two 2-cocycles for the $N = (8, 8)$ case. Some examples are investigated.

1 Introduction

Supersymmetric two dimensional nonlinear sigma models have important role in theoretical and mathematical physics such as their numerous string applications. Let us have a short bibliography for this subject. The relation between these theories and geometry of the target spaces have been studied about thirty five years ago [1]. The biHermitian geometry of the target spaces of the $N = 2$ extended supersymmetric sigma models was first realized in [2] (see also [3]). Then the extensions to more supersymmetries $N = 4$ and $N = 8$ have been investigated [4] (see also [5]). The sigma models with extended supersymmetry can only be defined on a restricted class of target manifolds, more supersymmetry implies more restriction on these geometries [5]. The extended supersymmetric sigma models on Lie group manifolds and also SUSY WZW models have been studied in [6]. The $N = 2$ and $N = 4$ extended superconformal field theories in two dimensions and also their correspondence with Manin triples have been investigated in [7] and [8]. Also there are some notes about $N = 8$ superconformal field theory in [7]. The algebraic study of $N = (2, 2)$ SUSY WZW models and also $N = (2, 2)$ SUSY sigma models on Lie groups (*algebraic biHermitian structures*) have been studied in [9] and [10], respectively.

In this paper, we try to obtain the algebraic structures of $N = (4, 4)$ and $N = (8, 8)$ SUSY sigma models on Lie groups (in general) and the algebraic structures of SUSY WZW models (especially).

The outline of the paper is as follows: in section two, we review the $N = (2, 2)$ SUSY sigma models on Lie groups and their *algebraic biHermitian structures* [10] as well as SUSY WZW models and their correspondence to Manin triples. Then in section three, we obtain the *algebraic bihypercomplex structures* for the $N = (4, 4)$ SUSY sigma models on Lie groups and specially for the $N = (4, 4)$ SUSY WZW models. We show their correspondence to Lie bialgebra with one 2-cocycle, at the end of this section we give an example. Finally in section four the algebraic structure of the $N = (8, 8)$ two dimensional SUSY sigma models on Lie group is investigated and show that for the $N = (8, 8)$ SUSY WZW models these algebraic structure is the Manin triples with two 2-cocycles, an example is given at the end of this section.

*e-mail: aali@azaruniv.edu

†Corresponding author. e-mail:rezaei-a@azaruniv.edu

2 $N = (2, 2)$ SUSY sigma models on Lie groups and SUSY WZW models

In this section, for self containing of the paper we will review briefly the geometric description of the $N = (2, 2)$ SUSY WZW and sigma-models on Lie groups [2]-[6] and their algebraic structures [9],[10]. We will use the $N = (1, 1)$ action to the description of $N = (2, 2)$ model; and impose extended supersymmetry on the superfields. With the knowledge that N supersymmetric sigma-models have N supersymmetric generator (Q_i) and $N - 1$ complex structures (J_i) on manifolds M such that for $N = (p, q)$ SUSY sigma-models in two-dimension then we will have p right-handed generators (Q_{i+}) and q left-handed generators (Q_{i-}) respectively, then $N = (1, 1)$ SUSY sigma model have one right-handed generators (Q_+) and one left-handed generators (Q_-) and the action on the manifold M is written as follows [2]:

$$S = \int d^2\sigma d^2\theta D_+ \Phi^\mu D_- \Phi^\nu (G_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)), \quad (1)$$

such that this action is invariant under the following supersymmetry transformation:

$$\delta_1(\epsilon)\Phi^\mu = i(\epsilon^+ Q_+ + \epsilon^- Q_-)\Phi^\mu, \quad (2)$$

where Φ^μ are $N = 1$ superfields; so that their bosonic parts are the coordinates of the manifold M . Further more the bosonic parts of the $G_{\mu\nu}(\Phi)$ and $B_{\mu\nu}(\Phi)$ are metric and antisymmetric tensors on M respectively. Note that in the above relations Q_\pm and D_\pm are supersymmetry generators and superderivative, respectively and ϵ^\pm are parameters of supersymmetry transformations. The above action has also invariant under the following extended supersymmetry transformation [2]:

$$\delta_2(\epsilon)\Phi^\mu = \epsilon^+ D_+ \Phi^\nu J_{+\nu}^\mu(\Phi) + \epsilon^- D_- \Phi^\nu J_{-\nu}^\mu(\Phi), \quad (3)$$

where $J_{\pm\nu}^\mu \in TM \otimes T^*M$. The consequence of invariance of the action (1) under the above transformations are the following conditions on $J_{\pm\sigma}^\rho$ [2]:

$$J_{\pm\lambda}^\mu J_{\pm\nu}^\lambda = -\delta_\nu^\mu, \quad (4)$$

$$J_{\pm\rho}^\mu G_{\mu\nu} = -G_{\mu\rho} J_{\pm\nu}^\mu, \quad (5)$$

$$\nabla_\rho^{(\pm)} J_{\pm\nu}^\mu = J_{\pm\nu,\rho}^\mu + \Gamma_{\rho\sigma}^{\pm\mu} J_{\pm\nu}^\sigma - \Gamma_{\rho\nu}^{\pm\sigma} J_{\pm\sigma}^\mu = 0, \quad (6)$$

where the extended connections $\Gamma_{\rho\sigma}^{\pm\mu}$ have the following forms:

$$\Gamma_{\rho\nu}^{\pm\mu} = \Gamma_{\rho\nu}^\mu \pm G^{\mu\sigma} H_{\sigma\rho\nu}, \quad (7)$$

such that

$$H_{\mu\rho\sigma} = \frac{1}{2}(B_{\mu\rho,\sigma} + B_{\rho\sigma,\mu} + B_{\sigma\mu,\rho}), \quad (8)$$

and $\Gamma_{\rho\nu}^\mu$ are Christofel symbols.

In order to have a closed supersymmetry algebra we must have the integrability condition on the complex structures (J_\pm) (4) as follows [2]:

$$N_{\mu\nu}^\rho(J_\pm) = J_{\pm\mu}^\gamma \partial_{[\gamma} J_{\pm\nu]}^\rho - J_{\pm\nu}^\gamma \partial_{[\gamma} J_{\pm\mu]}^\rho = 0. \quad (9)$$

In this manner the $N = (2, 2)$ SUSY structure of the sigma model on M is equivalent to existence of the biHermitian complex structure (J_\pm) on M (4),(5),(9) such that their covariant derivatives with respect to extended connection $\Gamma_{\rho\nu}^{\pm\mu}$ are equal to zero (6). If M is a Lie group G then in the non-coordinate bases, we have:

$$G_{\mu\nu} = L_\mu^A L_\nu^B G_{AB} = R_\mu^A R_\nu^B G_{AB}, \quad (10)$$

$$f_{AB}^C = L_\nu^C (L_\mu^A \partial_\mu L_\nu^B - L_\mu^B \partial_\mu L_\nu^A) = -R_\nu^C (R_\mu^A \partial_\mu R_\nu^B - R_\mu^B \partial_\mu R_\nu^A), \quad (11)$$

$$J_{-\nu}^\mu = L^\mu{}_A J^A{}_B L_\nu{}^B, \quad J_{+\nu}^\mu = R^\mu{}_A J^A{}_B R_\nu{}^B, \quad (12)$$

$$H_{\mu\nu\lambda} = L_\mu{}^A L_\nu{}^B L_\lambda{}^C, \quad H_{ABC} = R_\mu{}^A R_\nu{}^B R_\lambda{}^C H_{ABC}, \quad (13)$$

where G_{AB} is the ad-invariant nondegenerate metric and H_{ABC} is antisymmetric tensor on the Lie algebra \mathfrak{g} of the Lie group G . Note that $L_\mu{}^A(R_\mu{}^A)$ and $L^\mu{}_A(R^\mu{}_A)$ are components of left(right) invariant one-forms and their inverses on the Lie group G ; $f_{AB}{}^C$ are structure constants of the Lie algebra \mathfrak{g} and $J_B{}^A$ is an algebraic map $J : \mathfrak{g} \rightarrow \mathfrak{g}$ or *algebraic complex structure*. Now, using the above relations and the following relations for the covariant derivative of the left invariant veilbin [11]:

$$\nabla^\rho L^\eta{}_A = -\frac{1}{2}[f_A^{(\rho\eta)} + f_A^{\eta\rho} - T_A^{(\rho\eta)} - T_A^{\eta\rho} - L^{\eta B} \nabla^\rho G_{BA} + L^{\rho B} \nabla^\eta G_{AB} + L^\eta{}_A L^\rho{}_B \nabla_A G^{AB}], \quad (14)$$

then, we have the following algebraic relations, for the biHermitian geometry of the $N = (2, 2)$ SUSY sigma models [10]:

$$G\chi_A = (\chi_A G)^t, \quad (15)$$

$$J_C{}^B J_B{}^A = -\delta_C{}^A, \quad (16)$$

$$J_C{}^A G_{AB} J^B{}_D = G_{CD}, \quad (17)$$

$$H_{EFG} = J_E{}^A J_F{}^C H_{ACG} + J_G{}^A J_E{}^C H_{ACF} + J_F{}^A J_G{}^C H_{ACE}, \quad (18)$$

$$(H_A + \chi_A G)J = [(\chi_A G + H_A)J^t]^t, \quad (19)$$

where $(\chi_A)_B{}^C = -f_{AB}{}^C$ are the matrices in the adjoint representation and we have $(H_A)_{BC} = H_{ABC}$ for the matrices H_A . Note that relation (15) represents the ad-invariance of the Lie algebra metric G_{AB} . One can use relation (15)-(19) as a definition of *algebraic bi-Hermitian structure* on Lie algebra [10]; and calculate and also classify such structures on the Lie algebras [10]. For the $N = (2, 2)$ SUSY WZW models we have $H_{ABC} = f_{ABC}$; then (19) automatically satisfy and from (16) we obtain the determinant of J^2 is $(-1)^n$, i.e the dimension of the Lie algebra \mathfrak{g} (n) must be even and $J_B{}^A$ has eigenvalues $\pm i$. If we choose a basis $T_A = (T_a, T_{\bar{a}})$ for the Lie algebra \mathfrak{g} we will have [9]:

$$J = \begin{pmatrix} i\delta_b^a & 0 \\ 0 & -i\delta_{\bar{b}}^{\bar{a}} \end{pmatrix}, \quad (20)$$

where this form of J is satisfying in (18). In this basis according to (17) we must have the following form for G_{AB} :

$$G = \begin{pmatrix} 0 & g \\ g^t & 0 \end{pmatrix}, \quad (21)$$

where g is a $\frac{n}{2} \times \frac{n}{2}$ symmetric matrix. According to (18), we have $f_{abc} = 0$ and $f_{\bar{a}\bar{b}\bar{c}} = 0$, this means that $f_{ab}^c = f_{\bar{a}\bar{b}}^{\bar{c}} = 0$ i.e T_a and $T_{\bar{a}}$ form Lie subalgebras \mathfrak{g}_+ and \mathfrak{g}_- such that $(\mathfrak{g}_+, \mathfrak{g}_-)$ is a Lie bialgebra and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple [9]. The relation between Manin triples and $N = 2$ superconformal models (from the algebraic OPE point of view) was first pointed out in [7]. Also the relation of $N = (2, 2)$ WZW models and Manin triple (from the action point of view) was pointed in [9]. In [10] we have obtained all algebraic biHermitian structures related to four dimensional real Lie algebra. Let us consider a simple example for $N = (2, 2)$ SUSY WZW models correspond to the following non-Abelian four dimensional Manin triple $\mathbf{A}_{4,8}$ [10]:

$$\begin{aligned} [T_2, T_4] &= T_2, & [T_3, T_4] &= -T_3, & [T_2, T_3] &= T_1, \\ [T_{\bar{2}}, T_{\bar{4}}] &= T_{\bar{2}}, & [T_{\bar{3}}, T_{\bar{4}}] &= -T_{\bar{3}}, & [T_{\bar{2}}, T_{\bar{3}}] &= T_{\bar{1}}, \end{aligned} \quad (22)$$

$$G = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}. \quad (23)$$

3 N=(4,4) SUSY WZW and sigma models on Lie groups

As mentioned above, the correspondence between $N = 2$ and also $N = 4$ and $N = 8$ superconformal Kac-Mody algebra and Manin triples has been investigated in [7] and the Manin triples construction of $N = 4$ superconformal field theories has also investigated in [8], but up to now the algebraic structures of the $N = (4, 4)$ and $N = (8, 8)$ SUSY sigma models on Lie groups and also the $N = (4, 4)$ and $N = (8, 8)$ SUSY WZW models and their relations to Manin triples (from the action point of view) are not studied explicitly. Here, in this section we consider $N = (4, 4)$ case and in the next section the $N = (8, 8)$ case.

As in the previous section, we consider the $N = (1, 1)$ SUSY sigma model action (1) where invariant under transformation (2). Now we will consider for $N = (4, 4)$ case the invariance of that action under the following SUSY transformations [2],[3] (instead of (3)) and also $N = (4, 4)$ SUSY sigma model must be have four right-handed generators (Q_{+r}) and four left-handed generators (Q_{-r}) and three complex structures ($J_{\pm r}$):

$$\delta_{2r}(\epsilon)\Phi^\mu = \epsilon_r^+ D_+ \Phi^\nu J_{+r\nu}{}^\mu(\Phi) + \epsilon_r^- D_- \Phi^\nu J_{-r\nu}{}^\mu(\Phi), \quad r = 1, 2, 3, \quad (24)$$

such that the constrains on the complex structures are followed as [5]:

$$J_{\pm r\nu}{}^\lambda J_{\pm r\lambda}{}^\mu = -\delta_\nu^\mu, \quad (25)$$

$$J_{\pm r\nu}{}^\lambda J_{\pm s\lambda}{}^\mu = J_{\pm t\nu}{}^\lambda, \quad r \neq s \neq t = 1, 2, 3, \quad (26)$$

$$J_{\pm r\rho}{}^\mu G_{\mu\nu} = -G_{\mu\rho} J_{\pm r\nu}{}^\mu, \quad (27)$$

$$\nabla_\rho^{(\pm)} J_{\pm r\nu}{}^\mu = \partial_\rho J_{\pm r\nu}{}^\mu + \Gamma_{\rho\sigma}^{\pm\mu} J_{\pm r\nu}{}^\sigma - \Gamma_{\rho\nu}^{\pm\sigma} J_{\pm r\sigma}{}^\mu = 0, \quad (28)$$

where the closed characteristic of the algebra of SUSY transformations (i.e $[\delta_r^2(\epsilon_r), \delta_r^2(\epsilon_r)]$, and $[\delta_r^2(\epsilon_r), \delta_s^2(\epsilon_s)]$) have been consequences the following relations [5]:

$$J_{\pm r}{}^\lambda{}_{[\mu} \partial_\lambda J_{\pm r}{}^{\gamma}{}_{\nu]} - J_{\pm r}{}^\lambda{}_{[\mu} \partial_\lambda J_{\pm r}{}^{\gamma}{}_{\nu]} = 0, \quad (29)$$

$$J_{\pm r}{}^\mu{}_\lambda J_{\pm s}{}^\lambda{}_\nu + J_{\pm s}{}^\mu{}_\lambda J_{\pm r}{}^\lambda{}_\nu = 0, \quad r \neq s, \quad (30)$$

$$J_{\pm r}{}^\gamma{}_\lambda \partial_{[v} J_{\pm s}{}^\lambda{}_{\mu]} + J_{\pm r}{}^\lambda{}_{[\mu} \partial_\lambda J_{\pm s}{}^{\gamma}{}_{\nu]} + J_{\pm s}{}^\gamma{}_\lambda \partial_{[v} J_{\pm r}{}^\lambda{}_{\mu]} + J_{\pm s}{}^\lambda{}_{[\mu} \partial_\lambda J_{\pm r}{}^{\gamma}{}_{\nu]} = 0, \quad (31)$$

such that these are Nijenhuis-concomitant [12] for complex structures $J_{\pm r}$ ¹. When the background is a Lie group G then in non-coordinate bases ((10)-(13)) the geometrical relations (25)-(31) have the following algebraic forms:

$$J_{rC}{}^B J_{rB}{}^A = -\delta_C^A, \quad (32)$$

$$J_{rC}{}^B J_{sB}{}^A = J_{rC}{}^A, \quad (33)$$

$$J_{rC}{}^A G_{AB} J_{rD}{}^B = G_{CD}, \quad (34)$$

$$H_{EFG} = J_{rE}{}^A J_{rF}{}^C H_{ACG} + J_{rG}{}^A J_{rE}{}^C H_{ACF} + J_{rF}{}^A J_{rG}{}^C H_{ACE}, \quad (35)$$

$$(H_A + \chi_A G) J_r = [(\chi_A G + H_A) J^t]_r^t, \quad (36)$$

$$J_{sD}{}^B J_{rB}{}^A + J_{rD}{}^B J_{sB}{}^A = 0, \quad r \neq s, \quad (37)$$

$$\begin{aligned} & f_{B'A}{}^B J_{rB}{}^{C'} J_{sA'}{}^A + f_{A'B'}{}^A J_{rB}{}^{C'} J_{sA}{}^B + f_{AA'}{}^B J_{rB}{}^{C'} J_{sB'}{}^A + f_{AB}{}^{C'} J_{rA'}{}^A J_{sB'}{}^B \\ & + f_{B'A}{}^B J_{rA'}{}^A J_{sB}{}^{C'} + f_{BA}{}^{C'} J_{rB'}{}^A J_{sA'}{}^B + f_{AA'}{}^B J_{rB'}{}^A J_{sB}{}^{C'} + f_{A'B'}{}^B J_{rB}{}^A J_{sA}{}^{C'} = 0, \end{aligned} \quad (38)$$

In this way, relation (32)-(38) define the *algebraic bihypercomplex structures*² on the Lie algebra \mathfrak{g} , such that we have three algebraic complex structures J_r , ($r = 1, 2, 3$) where by use of (33) only two of them are independent i.e we have two algebraic independent complex structures (e.g J_1 and J_2). As for the $N = (2, 2)$ case for the

¹The Nijenhuis concomitant of J_r and J_s has the following form [12]:

$$N(I, J)^\lambda{}_{\mu\nu} = [I^\gamma{}_\mu \partial_\gamma J^\lambda{}_\nu - (\mu \longleftrightarrow \nu) - (I^\lambda{}_\gamma \partial_\mu J^\gamma{}_\nu - (\mu \longleftrightarrow \nu))] + (I \longleftrightarrow J)$$

²Similar to the name of bihypercomplex geometry [13].

$N = (4, 4)$ SUSY WZW models we have $H_{ABC} = f_{ABC}$ then relations (36) automatically satisfy and from (32), (34) and (35) one can obtain the following forms for J_1 , J_2 and G :

$$J_1 = \begin{pmatrix} i\delta_b^a & 0 \\ 0 & -i\delta_b^{\bar{a}} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & g \\ g^t & 0 \end{pmatrix}, \quad (39)$$

and

$$(J_2)_A{}^B = R_A{}^B = \begin{pmatrix} R_b^a & R_b^{\bar{a}} \\ R_{\bar{b}}^a & R_{\bar{b}}^{\bar{a}} \end{pmatrix}, \quad (40)$$

where we have the basis $T_A = \{T_a, T_{\bar{a}}\}$ for the Lie algebra \mathfrak{g} . Then from (37) one can obtain $R_b^a = R_{\bar{b}}^{\bar{a}} = 0$, and from (34) we obtain that $R^T = -R$, then from (32) we see that dimension of J_2 must be $4n$ where n is an integer number. So the dimension of Lie algebra \mathfrak{g} must be $4n$. Note that from (35) as for $N = (2, 2)$ case we see that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where \mathfrak{g}_+ and \mathfrak{g}_- are Lie subalgebras with basis $T_\Gamma = \{T_a, T_{\bar{a}}\}$ and $T'_\Gamma = \{T'_a, T'_{\bar{a}}\}$, $\bar{a}, a = 1, \dots, n$, such that the basis for \mathfrak{g} are now $T_A = \{T_\Gamma, T_{\bar{\Gamma}}\}$ and they form a Lie bialgebra. Now from (38) we have:

$$f_{AB}{}^D R_D{}^C + f_{DA}{}^C R_B{}^D - f_{BD}{}^C R_A{}^D = 0. \quad (41)$$

This means that we have a 2-cocycle. To show this we consider the definition of coboundary operator δ on an i -cochain γ on the Lie algebra \mathfrak{g} with values in the space M as follows [14]:

$$\delta\gamma(T_0, T_1, \dots, T_i) = \Sigma_{j=0}^i T_j \otimes (\gamma(T_0, \dots, \hat{T}_j, \dots, T_i)) + \Sigma_{j < k} (-1)^{j+k} \gamma([T_j, T_k], T_0, \dots, \hat{T}_j, \dots, T_k, \dots, T_i), \quad (42)$$

$\forall T_A \in \mathfrak{g}$. The 2-cochain γ is 2-cocycle when $\delta\gamma = 0$. Now for the case that $M = \mathbb{C}$ we have:

$$\begin{aligned} & -\delta\gamma(T_0, T_1, T_2) + T_0 \otimes (\gamma(T_1, T_2)) + T_1 \otimes (\gamma(T_0, T_2)) + T_2 \otimes (\gamma(T_0, T_1)) \\ & -\gamma([T_0, T_1], T_2) + \gamma([T_0, T_2], T_1) - \gamma([T_1, T_2], T_0) = 0. \end{aligned} \quad (43)$$

Using the following form for the 2-cochain:

$$\gamma(T_A, T_B) = (R_{AB})^{\Gamma\Lambda} T_\Gamma \otimes T_\Lambda + (R_{AB})^{\Gamma\bar{\Lambda}} T_\Gamma \otimes T_{\bar{\Lambda}} + (R_{AB})^{\bar{\Gamma}\Lambda} T_{\bar{\Gamma}} \otimes T_\Lambda + (R_{AB})^{\bar{\Gamma}\bar{\Lambda}} T_{\bar{\Gamma}} \otimes T_{\bar{\Lambda}}, \quad (44)$$

in (43) after some calculation one can obtain (41). In this way the algebraic structure of $N = (4, 4)$ WZW models is also Lie bialgebra as for the $N = (2, 2)$ WZW models with this difference that for the $N = (4, 4)$ case, we have Lie bialgebra with a 2-cocycle, such that the independence algebraic complex structures (J_1, J_2) are anticommute (37).

As for the $N = (2, 2)$ case we consider the non-Abelian four dimensional Manin triple $\mathbf{A}_{4,8}$. Now in this case ($N = (4, 4)$) we have the following forms for the metric G and complex structures J_1 and J_2 .

$$\begin{aligned} G &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ J_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \end{aligned} \quad (45)$$

with the following 2-cocycle:

$$R = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix} \quad (46)$$

where I is 2×2 unit matrix.

4 N=(8,8) WZW and sigma models on Lie groups

Now, as for the $N = (4,4)$ case we consider the action (1) again; such that this action is a invariant under SUSY transformation (2) as well as under the following second SUSY transformations [5]:

$$\delta_{2r}(\epsilon)\Phi^\mu = \epsilon_r^+ D_+ \Phi^\nu J_{+r\nu}{}^\mu(\Phi) + \epsilon_r^- D_- \Phi^\nu J_{-r\nu}{}^\mu(\Phi), \quad r = 1, \dots, 7 \quad (47)$$

where for these transformations we have fourteen $J_{\pm r}$ geometric complex structures. As for the $N = (4,4)$ case from the invariance of the action (1) under transformation (47) and also closed characteristic of the algebra of transformations one can obtain again relations similar to (25)-(31) with $(r = 1, \dots, 7)$ [5] and also the same algebraic relations (32)-(38). For this case from (34) we have the following relations among algebraic complex structures

$$J_1 J_2 J_3 J_4 J_5 J_6 = J_7, \quad J_2 J_3 J_4 J_5 = J_6, \quad J_3 J_4 = J_5, \quad J_1 J_2 = J_3, \quad (48)$$

therefore only three of them (e.g J_1 , J_2 and J_4) are independent. As for the $N = (4,4)$ case for $N = (8,8)$ SUSY WZW we obtain the following forms for the complex structures J_1 , J_2 and J_3 and also for G :

$$J_1 = \begin{pmatrix} i\delta_6^a & 0 \\ 0 & -i\delta_6^{\bar{a}} \end{pmatrix}, \quad G = \begin{pmatrix} 0 & g \\ g^t & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & R_{1b}^{\bar{a}} \\ R_{1\bar{b}}^a & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & R_{2b}^{\bar{a}} \\ R_{2\bar{b}}^a & 0 \end{pmatrix}, \quad (49)$$

where in this case from (32) we conclude that the dimension of the algebra \mathfrak{g} must be $8n$ with n is an integer and relation (35) for J_2 reduce the Lie bialgebra structures with Lie subalgebras \mathfrak{g}_+ and \mathfrak{g}_- with dimension $4n$. In this case relation (38) reduce to the following relations:

$$f_{AB}{}^D R_{1D}{}^C + f_{DA}{}^C R_{1B}{}^D - f_{BD}{}^C R_{1A}{}^D = 0, \quad (50)$$

$$f_{AB}{}^D R_{2D}{}^C + f_{DA}{}^C R_{2B}{}^D - f_{BD}{}^C R_{2A}{}^D = 0, \quad (51)$$

i.e the algebraic structures of $N = (8,8)$ WZW models are Lie bialgebras with two 2-cocycles and three algebraic complex structure J_1 , J_2 and J_3 where anticommute under (37).

As an example, consider a four dimensional complex Lie algebra L_9 with the following commutations relations [15]:

$$[T_1, T_2] = T_2, \quad [T_3, T_4] = T_4. \quad (52)$$

one of the dual Lie algebra for the above Lie algebra is \bar{L}_9 that satisfy in the following mixed Jacobi identities:

$$f_{mk}{}^i \bar{f}^{jm}{}_l - f_{ml}{}^i \bar{f}^{jm}{}_k - f_{mk}{}^j \bar{f}^{im}{}_l + f_{ml}{}^j \bar{f}^{im}{}_k = f_{kl}{}^m \bar{f}^{ij}{}_m. \quad (53)$$

with following commutation relations:

$$[T_1, T_2] = T_2, \quad [T_3, T_4] = T_4. \quad (54)$$

Now, for this 8 dimensional Lie algebra \mathfrak{g} we have obtained the following algebraic complex structures J_1 , J_2 and J_3 and metric G :

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J_1 = \begin{pmatrix} -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \end{pmatrix}, \quad (55)$$

with the following two 2-cocycles:

$$R_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (56)$$

5 conclusion

We have obtained the algebraic structures of the $N = (4, 4)$ and $N = (8, 8)$ SUSY two dimensional sigma models on Lie groups (in general) and the $N = (4, 4)$ and $N = (8, 8)$ SUSY WZW models (in special). We have shown that as for the $N = (2, 2)$ case these structures correspond to the Lie bialgebra structures with one 2-cocycle for the $N = (4, 4)$ and two 2-cocycles for the $N = (8, 8)$ case. As an open problem in the forthcoming work one can use the relations of algebraic structures for $N = (4, 4)$ and $N = (8, 8)$ ((32)-(38)) to obtain and classify all these structures on low dimensional Lie algebra as for the $N = (2, 2)$ case [10].

Acknowledgment: We would like to thanks from F. Darabi for carefully reading the manuscript and useful comments.

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